A Unified Branch-and-Benders-Cut for Two-Stage Stochastic Mixed-Integer Programs

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1 Introduction

Two-stage stochastic mixed-integer programs (2SMIPs) are optimization problems under uncertainty with variables partitioned into (i) *first-stage* decisions taken before the realization of the random events and (ii) *second-stage* decisions taken after the occurrence of a specific scenario. Let $\widetilde{\omega}$ be a random vector that characterizes possible outcomes for the stochastic parameters, and that is defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and let $\mathbb{E}[\cdot]$ denote the usual mathematical expectation operator, a standard formulation for 2SMIPs is:

$$
\begin{aligned}
\min \quad & c^T \cdot x + \mathbb{E}[h(x, \widetilde{\omega})] && (1^{st} \text{ stage}) \\
s.t. \quad & Ax \geq b && \text{(1a)} \\
& x \in \mathbb{X},
\end{aligned}
$$

where, for a given scenario ω of $\widetilde{\omega}$, $h(x, \omega)$ is defined as:

$$
\begin{aligned}
\min \ h(x, \omega) = \quad & q_\omega^T \cdot y_\omega && (2^{nd} \text{ stage}) \\
s.t. \quad & T_\omega \cdot x + W_\omega \cdot y_\omega \geq h_\omega && \text{(2a)} \\
& y_\omega \in \mathbb{Y}.
\end{aligned}
$$

$\mathbb{X} \subseteq \mathbb{R}_+^{n_1}$ and $\mathbb{Y} \subseteq \mathbb{R}_+^{n_2}$ define the domains of the first-stage variables, x, and the second-stage variables, y_ω, respectively. Parameters $c \in \mathbb{R}^{n_1}, A \in \mathbb{R}^{m_1 \times n_1}, b \in \mathbb{R}^{m_1}$ are known in advance, while $q_\omega \in \mathbb{R}^{n_2}$, $T_\omega \in \mathbb{R}^{m_2 \times n_1}$, $W_\omega \in \mathbb{R}^{m_2 \times n_2}$, $h_\omega \in \mathbb{R}^{m_2}$ are related to scenario ω. $h(x, \omega)$ is the optimum value of the scenario subproblem parametrized with x and ω. The objective is to minimize the first-stage cost and the expected cost of the recourse decisions.

By creating copies of the second-stage variables for each scenario, one can formulate 2SMIPs in a compact form referred to as the *deterministic equivalent formulation* (DEF). Note that the number of variables of the DEF increases with the number of scenarios considered. As such, while a general-purpose MIP solver can solve the DEF, it is expected to be inefficient for large-sized instances. Alternatively, as the DEF constraint matrix exhibits a *block-angular structure*, 2SMIPs are well-suited to Benders-decomposition algorithms which leverage scenario decomposition. As the Benders algorithm's convergence relies on cuts obtained by applying standard linear programming duality theory to the subproblem, it can only be applied to 2SMIPs with continuous recourse. To the best of our knowledge, and despite the importance of this class of problem, the literature solely includes two exact algorithmic strategies ([1, 2]) for generic 2SMIPs, i.e., 2SMIPs with general mixed-integer variables in both stages and no restriction on the stochastic parameters.

2 Unified Branch-and-Benders-Cut

We introduce the Unified Branch-and-Benders-Cut (UB&BC), a two-phase generic approach for solving 2SMIPs with discrete scenario subproblems.

The first phase applies a modified Branch-and-Benders-Cut to the master problem. As an integer solution is found in the branch-and-bound tree, scenario subproblem LP relaxations are used as separation problems to generate classical Benders cuts. These cuts may eliminate the current integer solution. As the Benders cut generation yields a fractional master solution, a new variable to branch on is chosen, and the branch-and-bound search resumes. If the Benders cut-generation terminates with a candidate integer solution $\widehat{x}$, we avoid solving MIP scenario subproblems to alleviate the computational burden. Instead, we determine bounds on $opt(\widehat{x})$, i.e., the objective function value obtained by combining $\widehat{x}$ with its MIP scenario subproblem optimal solutions. Specifically, we derive a lower and an upper bound on $opt(\widehat{x})$, which enable updating the best lower and upper bounds, lb^* and ub^*, and potentially eliminating solution $\widehat{x}$. If $\widehat{x}$ cannot be eliminated, it is retained as an *open* solution. Classically, the search converges if the gap between lb^* and ub^* falls within a predefined optimality tolerance. However, because of (i) the Benders cuts based on scenario subproblem linear relaxations and (ii) heuristic upper bounds, the search may end with a gap between lb^* and ub^* greater than the predefined optimality tolerance. In that case, the Branch-and-Benders-Cut *modified* rules for fathoming nodes guarantee that the global optimal solution is among the *open* solutions retained.

In the second phase, *open* solutions are successively evaluated by solving the corresponding MIP scenario subproblems until the optimal solution to the original problem is found. The UB&BC applies as the DEF constraint matrix exhibits a general structure. However, UB&BC may be less effective in this case since scenario decomposition cannot be leveraged.

3 Experiments

We address the two-stage stochastic traveling salesman with outsourcing (2TSP), a problem that finds applications in the repair and maintenance industry. The first stage aims to select customers $i \in N$ to be served by the vehicle if they request a service in the second stage. On the other hand, unselected customers who happen to request a service in the second stage are outsourced to a third party, incurring an extra fixed cost. We consider scenarios $\omega \in \Omega$, each associated with a probability p_ω. In each scenario, the binary parameter h_i^ω represents whether customer $i \in N$ has a request. For a given scenario, the second-stage problem aims to determine a tour with minimal cost that visits the selected customers requesting a service. Let $E = \{ (i,j) \mid i < j, (i,j) \in N^2 \}$ be the set of all edges forming a complete graph between the nodes $i \in N$. The 2TSP decision variables are: (i) x_i, binary variables that equal 1 iff customer i is selected, and (ii) y_{ij}^ω, binary variables that equal 1 iff edge (i,j) is used in scenario ω.

We generated a set of 200 2TSP instances, with 50 to 500 scenarios, that extend the classic TSPLib instances and are accessible at: `http://elib.zib.de/pub/mp-testdata/tsp/tsplib/tsp/index.html`. We have solved these instances with different methods, namely, CPLEX default branch-and-cut solver and multiple versions of UB&BC that vary according to (i) the TSP heuristic used as an upper-bounding procedure for the scenario subproblems, and (ii) the master problem formulation. The time limit is set to 6 hours for all methods. Overall, CPLEX could solve the DEF to optimality for only 12 instances, against 154 instances for the most effective version of UB&BC.

References

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