Box-total dual integrality of the perfect matching polytope

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1 Introduction

The *perfect matching polytope* is the convex hull of the characteristic vectors of all perfect matchings of a graph. A graph is *matching covered* if each of its edges belongs to a perfect matching. These graphs have been introduced by Edmonds, Pulleyblank, and Lovász in [6]. A matching covered graph G is *solid* if it has no separating cuts, where a cut $\delta(X)$ is *separating* if G/X and $G/(V \setminus X)$ are also matching covered.

A rational linear system is totally dual integral (TDI) if for every integer linear function for which the optimum is finite the associated dual problem has an integer optimal solution. A TDI system is box-TDI if adding any rational bounds on the variables preserves its TDIness. Box-TDI systems are systems that yield strong min-max relations such as the one involved in the Max Flow-Min Cut Theorem of Ford and Fulkerson. A polyhedron is box-TDI if it can be described by a box-TDI system. In brief, TDI systems provide sufficient conditions for the integrality of a polyhedron, while box-TDIness guarantees it even when adding box constraints, which can be viewed as introducing constrained capacity.

Box-totally dual integral systems and polyhedra received a lot of attention from the combinatorial optimization community around the 80's. A renewed interest appeared in the last decade and since then many deep results appeared involving such systems (see for exemple [2], [4], and [7]).

Ding et al. [5] characterized the box-TDIness of the matching polytope. Though the perfect matching polytope is a face of the latter, their result does not characterize the box-TDIness of the perfect matching polytope.

2 Contributions

We extend the work of de Carvalho et al. [3], by characterizing the perfect matching polytope of solid matching covered graphs.

Theorem 1. *The perfect matching polytope of a solid matching covered graph is the intersection of its affine hull with the nonnegative orthant.*

Then, combining a characterization of box-TDI polyhedra of Chervet et al. [2] with a result on incidence matrices of graphs [1], we prove the following.

Corollary 2. *The perfect matching polytope of a matching covered graph is box-TDI if and only if its affine hull is also box-TDI.*

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