Formal-simplex algorithm in linear programming

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1 Introduction

The simplex algorithm pivots from basic feasible solution (BFS) to another BFS attempting to reach a vertex whose reduced cost vector is non-positive. This process is achieved by improving the objective value in each step, avoiding in consequence to browse all BFS-s whose number is in general extremely large. In linear programming, it is sometimes easy to initialize the simplex algorithm by a basic solution that looks optimal (the related reduced cost is non-positive) but which is not feasible. In this case, the dual-simplex algorithm is applicable. This algorithm works towards feasibility while maintaining the optimality condition. In this paper, we propose another alternative, different of dual-simplex algorithm, that we agree to call formal-simplex algorithm. Thanks to the notion of formal tableau, developed in [3], we prove that the dual-simplex algorithm is actually the simplex algorithm with Dantzig's pivot rule ran on the formal tableau. Since there are many new pivot rules developed in the literature that are more efficient than Dantzig one's, our approach consists in applying simplex algorithm with an appropriate pivot rule on the formal tableau. To get the primal optimal solution, we apply the formal tableau on the last iteration. This action is justified since the formal tableau of the formal tableau is precisely the primal simplex tableau.

2 Formal-simplex algorithm

2.1 Formal tableau

Let us consider a linear program (LP) written in standard form:

$$\begin{cases} \max \text{ (or } \min) \ [Z(y) = c^* y] \\ My = b \\ y \geq 0_{\mathbb{R}^n}, \end{cases}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $M \in \mathcal{M}_{m,n}(\mathbb{R})$; with $m < n$ and $\text{rank}(M) = m$. The symbol * stands for the transpose operator. After possible rearranging the columns of M, let $[B \ \ N] = M$ where B is a $m \times m$ invertible matrix; called basic matrix; and N a $m \times (n-m)$ matrix. The simplex tableau in condensed form related to the basis B is as follows:

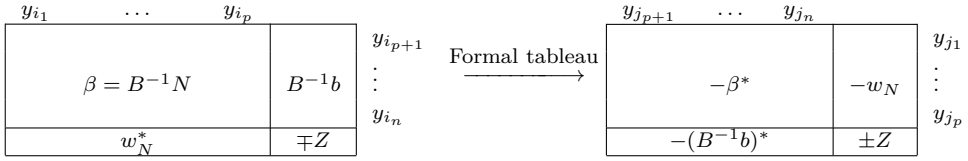
y_{j_1} $\cdots$ y_{j_p}		
$B^{-1}N$	$B^{-1}b$	y_{i_1} $\vdots$ y_{i_m}
$w_N^* = \pm[c_N^* - c_B^* B^{-1} N]$	$\mp Z$	

where $Z = Z\binom{B^{-1}b}{0_N} = c_B^* B^{-1} b$ and $p = n - m$.

The row vector $w_N^* = \pm[c_N^* - c_B^* B^{-1} N]$ is called the reduced cost vector associated to B. the $\pm$ symbol depends on the nature of the problem: it is $+$ if the studied (LP) is of type "max"

and it is $-$ for type "min". If $B^{-1}b \geqslant 0_{\mathbb{R}^m}$, the basis B is called feasible and the solution $\begin{pmatrix} B^{-1}b \\ 0_{\mathbb{R}^p} \end{pmatrix}$ is called the BFS associated to B. In this case, the condition $w_N^* \leqslant 0_N^*$ ensures the optimality of this BFS, otherwise a positive entry w_s of w_N^* is selected and the corresponding nonbasic variable y_{j_s} is permuted with the basic variable y_{i_r} where $r = \text{argmin}\{\frac{(B^{-1}b)_i}{(B^{-1}N_s)_i} \mid (B^{-1}N_s)_i > 0\}$. In that way, the new basis remains feasible and the objective value is enhanced: $Z \leftarrow Z \pm \frac{(B^{-1}b)_r}{(B^{-1}N_s)_r} w_s$. This process describes a simplex iteration and in case of multiple positive entries in w_N^*, the choice of w_s is not unique. The way, fixing which positive coefficient w_s must be selected, is called pivoting rule. The most known is the Dantzig's pivot rule, by reference to George Dantzig, it chooses systematically the most positive reduced cost [1].

In [3], Nabli introduced the notion of formal tableau. For each simplex tableau, a new tableau which is actually a simplex tableau for the dual problem, is defined. It is called formal tableau and it is obtained through the following correspondence:



The basic variables $y_{i_{p+1}}, \ldots, y_{i_n}$ are transformed into nonbasic but with an indexation defined by the following congruence relation:

$$j_l \equiv i_l + m \ \ [n] \tag{1}$$

For the nonbasic variables $y_{i_1}, \ldots, y_{i_p}$, they must be transposed to basic variables in the formal tableau with the same indexation rule (1).

Theorem 2.1 *The dual-simplex algorithm is exactly the simplex algorithm applied on the formal tableau with Dantzig's pivot rule.*

Proof. The right hand side $B^{-1}b$ in the primal tableau is transformed to the reduced cost vector with opposite sign in the formal tableau. As the dual-simplex algorithm starts by seeking the most negative entry in $B^{-1}b$, that corresponds, up to a sign, to the most positive reduced cost in the formal tableau. So, the exiting variable of the dual-simplex algorithm is exactly the entering variable in the formal tableau using the Dantzig's pivot rule. On the other hand, the entering variable for the dual-simplex algorithm is actually the exiting variable for the formal tableau since $\frac{a}{b} = \frac{-a}{-b}$ for all reals a and $b \neq 0$. □

By analogy to the duality in linear programming, the notion of the formal tableau satisfies a fundamental property stated in the following theorem.

Theorem 2.2

The formal tableau of the formal tableau is the primal tableau.

Proof. The formal tableau has p basic variables and then the indexation of the formal tableau of the formal tableau is:

$$k_l \equiv j_l + p \equiv i_l + p + m \ \ \text{mod}(n)$$

Since $p + m = n$, we obtain $k_l = i_l$. On the other hand, it is clear that for any matrix A, we have $-(-(A)^*)^* = A$. □

2.2 Algorithm description

The formal-simplex algorithm consists in running the simplex algorithm with an appropriate pivot rule on the formal tableau and in considering at the end the formal tableau of the last tableau to get the primal solution. The word "appropriate" can be defined by the user according to the considered LP. Our approach can be recapitulated in the algorithm below.

The consequences of the formal-tableau concept are multiple:

Algorithm 1 Formal-Simplex Algorithm

Require: A dual feasible basis B
Ensure: Resolution of the primal
  Construct the formal tableau related to B
  Run the simplex algorithm with an appropriate pivot rule on this formal tableau
  if Optimal solution **then**
    Consider the formal tableau of the last tableau to get the primal optimal solution
  end if
  if Unbounded solution **then**
    The primal is unfeasible
  end if

1. No need to implement two codes: one for the simplex and another for the dual-simplex.
2. No need to develop new pivoting rules for the dual-simplex algorithm.
3. For each considered (LP), we have two choices: either apply the simplex algorithm on the primal tableau or apply it on the formal tableau and consider at the end the formal tableau. Even if the initial basis is neither feasible nor dual-feasible, the nonfeasible basis method, developed in [3, 4], is an alternative for such a situation [5].

Many examples are performed in the next section, the results show the advantage of our approach in number of iterations and computational time compared with the dual-simplex algorithm.

3 Results of Comparison

In order to show the advantages of our approach, we compare it with the dual-simplex algorithm. For this purpose, we consider many examples all of the following type:

$$\begin{cases} \min[\psi(v) = c^* v] \\ Av \geqslant b \\ v \geqslant 0_{\mathbb{R}^p}, \end{cases}$$

where A is $m \times p$ matrix with Gaussian entries, c and b are Poissonian vectors of parameter $\mu = 1$ and $\mu \times p$ respectively. The number of iterations required by the dual-simplex (DS) and Formal-simplex (FSSE) algorithms are reported in Table 1. Since the entries of the considered linear programs are random, each algorithm is ran 10 times on each couple (m, p) and Table 1 displays the mean value of iteration numbers.

(m,p)	$(50,75)$	$(100,150)$	$(150,225)$	$(200,300)$	$(300,450)$	$(400,600)$	$\sum$
DS	113	416	787	1365	3151	5849	11681
FSSE	71	191	276	428	727	1334	3027

TAB. 1: Number of iterations for different values of (m, p)

In this paper, we adopt the steepest-edge pivot rule developed in [2]. The gain of our approach (FSSE), compared with the dual-simplex algorithm (DS), is substantial: 285% ($11681/3027 \simeq 3.859$). We also observe that the larger (m, p) is, the higher the gain.

Table 2 gives the CPU time in second for different examples when the latter are performed on a same machine through the software Matlab 2015. As predicted, These results show that the formal-simplex algorithm (FSSE) is better than the dual-simplex algorithm (DS). Precisely, (FSSE) is overall 3.6 more speedy than (DS) since we have $174.89/48.38 \simeq 3.615$. The gain in

(m,p)	$(50,75)$	$(100,150)$	$(150,225)$	$(200,300)$	$(300,450)$	$(400,600)$	$\sum$
DS	0.08	0.47	2.15	5.65	35.68	130.86	174.89
FSSE	0.02	0.13	0.48	1.55	9.96	36.24	48.38

TAB. 2: CPU time in second for different values of (m,p)

CPU time is slightly smaller than the gain in number of iterations. This is due to the additional cost required by the steepest-edge rule in the entering variable selection.

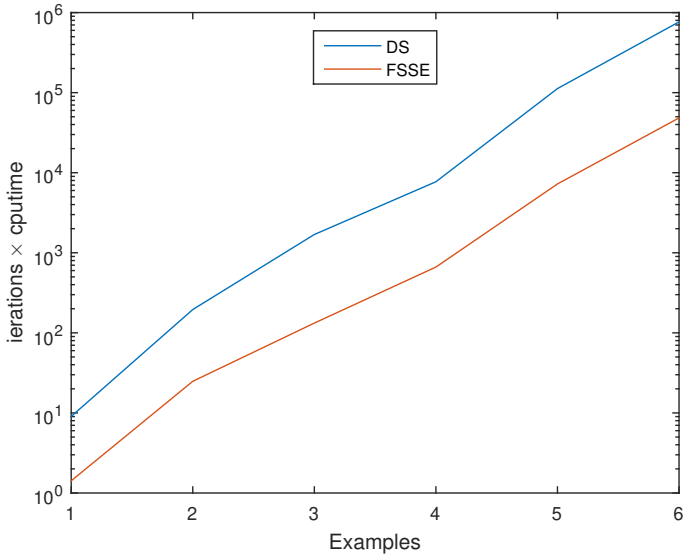


FIG. 1: Evolution of the product "iterations × cputime"

Figure 1 shows the behaviour of the product between the involved number of iterations and the cpu-time according to the different values (m,p), that are indexed from 1 to 6. We observe that the gap between the two curves (DS) and (FSSE) enlarges when (m,p) grows. This confirms our statement "the larger (m,p) is, the higher the gain".

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