Survivable Ring Star Problem under the failure of two hubs

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1 Introduction

Given a set V of nodes, the Ring Star Problem (RSP) considers selecting and linking some nodes (hubs) to form a ring and assigning remaining vertices (terminals) to nodes in the ring to create a star network. RSP aims to minimize the overall cost of designing a ring and star network, including the costs of selecting and connecting hubs in a ring and connecting terminals to hubs as a star. Xu et al. [1] first investigated RSP as a variation of the Steiner Tree problem. Then, Labbé *et al.* [4] presented a strengthened formulation and branch-and-cut approach. Since then, Karaşan *et al.* [2] considered RSP given that the links are disrupted. However, Khamphousone et al. [3] is the only known work on node failure in RSP settings. In this work, we consider the Survivable Ring Star Problem (2-S-RSP), in which we want to conserve the ring-star topology when, at most, two hubs in the ring can fail at a time, following the work of Khamphousone et al. [3].

2 Problem formulation

To define the 2-S-RSP, the following notations are introduced. A complete mixed graph denoted as $G = (V, E \cup A)$ is given as input, where V represents the vertex set, E is the edge set ($E = \{(i,j) \mid (i,j) \in V^2, i < j\}$), and A is the arc set ($A = \{(i,j) \mid (i,j) \in V^2, i \neq j\}$). A node is *certain* if it cannot fail when chosen as a hub, otherwise *uncertain*. We denote the given subset of uncertain nodes as $\widetilde{V} \subseteq V$. Node 1 serves as the *depot*, which is certain and always in the ring. The ring includes hubs, while non-hubs are referred to as terminals within the star. Each terminal is connected to one or more hubs in the ring.

The variable x_{ij} takes a binary value of 1 only when nodes i and j are neighboring in the ring. Similarly, the binary variable y_{ij} is set to 1 if terminal i is linked to hub j, and 0 otherwise. Additionally, y_{ii} is 1 only when node i serves as a hub. The binary variable x'_{ij} is set to 1 when edge (i,j) is used as a backup to handle hub failures in the ring. The set $\mathbb{B}$ represents the binary set $\{0,1\}$. We consider four distinct types of costs : ring cost r_{ij}, hub opening cost o_i, the cost associated with connecting terminals (star cost) s_{ij}, and the cost of establishing backup edges r'_{ij}. We formulate 2-S-RSP as the following integer linear program :

$$\text{Minimize} \sum_{(i,j)\in E} (r_{ij}x_{ij} + r'_{ij}x'_{ij}) + \sum_{i\in V} o_i y_{ii} + \sum_{(i,j)\in A} s_{ij}y_{ij}$$

$$\sum_{j\in V, i<j} x_{ij} + \sum_{j\in V, i>j} x_{ji} = 2y_{ii}, \quad \forall i \in V \tag{1}$$

$$\sum_{i\in S} \sum_{j\in S\setminus\{i\}} x_{ij} \leq |S| - 1, \quad \forall S \subseteq V, 1 \notin S \tag{2}$$

$$\sum_{i\in S} \sum_{j\in S\setminus\{i\}} x_{ij} \leq |S| - y_{kk}, \quad \forall S \subseteq V, 1 \in S, k \in V\setminus S \tag{3}$$

$$\sum_{j\in V\setminus(\widetilde{V}\cup\{i\})} 3y_{ij} + \sum_{j\in\widetilde{V}\setminus\{i\}} y_{ij} = 3(1-y_{ii}), \quad \forall i \in V \tag{4}$$

$$y_{11} = 1 \tag{5}$$

$$\sum_{(i,j)\in E} x_{ij} \geq 3 + \sigma \tag{6}$$

$$\sigma \geq y_{ii} + y_{jj}, \quad \forall i \in \widetilde{V}, j \in \widetilde{V}, i \neq j \tag{7}$$

$$x_{ik} + x_{kt} + x_{tj} \leq 2 + x'_{ij}, \quad \forall (k,t,i,j) \in \widetilde{J} \tag{8}$$

$$x_{ik} + x_{kj} \leq 1 + x'_{ij}, \quad \forall (i,j) \in V^2, k \in \widetilde{V}, i < j, k \neq t \tag{9}$$

$$y_{ij} \leq y_{jj}, \quad \forall (i,j) \in A \tag{10}$$

$$x_{ij}, x'_{ij} \in \mathbb{B}, \forall (i,j) \in E \quad y_{ij} \in \mathbb{B}, \forall (i,j) \in V^2$$

$$\sigma \in \mathbb{Z}_+$$

The objective function of this problem is to minimize the total cost, which includes the costs of connecting hubs in a ring, establishing backup edges, opening hubs, and building arcs in a star. The ring is connected using constraints (1), requiring each hub to connect to two others to complete a ring. Constraints (4) require each terminal to be connected to either one specific hub in $V\setminus\widetilde{V}$ or three uncertain hubs in $\widetilde{V}$. Constraint (5) imposes depot 1 to be a hub in the ring. Inequalities (6) and (7) guarantee a minimum ring size of four if $|\widetilde{V}| = 1$ and five when $|\widetilde{V}| \geq 2$. A three-ring size is required otherwise. Constraints (8) ensure a backup edge is created to handle the case where two adjacent uncertain hubs are failing. We denote $\widetilde{J} = \{(k,t,i,j) \mid (i,j) \in V^2, (k,t) \in \widetilde{V}^2, i < j; i,j,k,t \text{ are pairwise distinct}\}$. Constraints (9) guarantee a backup edge for each uncertain hub, and constraints (10) restrict terminal connections to hubs.

3 Contributions and initial results

In addition to the integer linear program (ILP) formulation of 2-S-RSP, we propose an accelerated Branch-and-Benders-cut (BBC) approach by dividing the ILP into a master problem and independent Benders subproblems. We also introduce valid inequalities, that make the linear programming relaxation of those subproblems integral. We then present an algorithm to compute the dual solutions and to separate Benders optimality cuts. Three acceleration techniques are also proposed : an instance transformation, which produces an equivalent instance but with more weight in the master problem, valid inequalities in the master problem, and multi-Benders cuts, which consist of decomposing the Benders subproblem into independent sub-subproblems, leading to multiple and less dense Benders optimality cuts.

Using TSPLIB and randomly generated instances, we compare the proposed Branch-and-Benders-cut method with the ILP formulation. The results show that the former method performs better with many nodes, while the latter has memory issues.

Références

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