Scaling-up exact methods for solving the discrete bilevel network design problem under user equilibrium

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1 Problem context and motivation

We address the bilevel discrete network design problem (DNDP) in transportation [4]. Given a transportation network and a budget, the DNDP consists of determining the optimal subset of links to be added to minimize congestion effects. Congestion is measured using traffic assignment where link travel times are modeled as convex flow-dependent functions and network users are represented as utility-maximizing agents making selfish route choice decisions. The collective choice of users can be formulated as a Nash equilibrium problem known as a Wardropian equilibrium [7]. The DNDP admits a natural bilevel optimization where the leader represents the network manager and the follower is a parameterized traffic assignment problem (TAP). The TAP is a convex optimization problem whose optimality conditions require that, for each commodity, all paths used at user equilibrium (UE) have minimum and equal travel time. Embedding UE conditions in a bilevel optimization formulation via path-based variables requires handling path set enumeration. While column generation techniques for (integer-) linear programs are well-developed, this is not the case for nonlinear programs. This has led researchers to develop link-based formulations amenable to mixed-integer nonlinear programming approaches [3, 6, 2]. However, computational studies have found that even after approximating link travel time functions via piecewise linear functions, exact algorithms struggle to solve the DNDP to global optimality on medium-size networks with more than 20 candidate links [5]. This study introduces a novel path-based approach to solve the DNDP based on Lagrangian relaxation that overcomes path enumeration challenges by leveraging the scalability of TAP algorithms.

2 DNDP formulation and solution method

The DNDP is defined on a network with a node set $\mathcal{N}$ and link set $\mathcal{A}$ as a multi-commodity network flow problem with nonlinear link travel time functions. Let $\mathcal{W}$ be the set of origin-destination (OD) pair in the network and let d_{rs} be the travel demand of OD pair $(r,s) \in \mathcal{W}$. Let Π_{rs} be the set of paths connecting OD pair $(r,s) \in \mathcal{W}$ and let h_π be the flow on path $\pi \in \Pi_{rs}$. Let x_{ij} be the total flow on link $(i,j) \in \mathcal{A}$. We denote $[\delta_{ij}^\pi]$ the link-path incidence matrix. Let $t_{ij}(\cdot)$ be a positive and increasing convex function representing the travel time on link $(i,j) \in \mathcal{A}$. Let $\mathcal{A}_1$ be the set of existing links and let $\mathcal{A}_2$ be the set of candidate links to improve the network, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. For each link $(i,j) \in \mathcal{A}_2$, let g_{ij} be the cost of adding this link to the network and let $y_{ij} \in \{0,1\}$ be the variable representing this choice. The impact of the leader decisions in the follower problem is achieved through the linking constraints

$x_{ij} \leq y_{ij} D$ wherein $D = \sum_{(r,s)\in W} d_{rs}$ is the total demand in the network. The follower is the Beckmann formulation of the TAP parameterized by leader variables $\boldsymbol{y}$ [1] :

$$\min_{\boldsymbol{x},\boldsymbol{h}} \quad \sum_{(i,j)\in\mathcal{A}} \int_0^{x_{ij}} t_{ij}(v)dv \tag{1a}$$

$$\text{s.t.} \quad \sum_{\pi\in\Pi_{rs}} h_\pi = d_{rs} \qquad \forall (r,s) \in \mathcal{W} \tag{1b}$$

$$\sum_{\pi\in\Pi} h_\pi \delta_{ij}^\pi = x_{ij} \qquad \forall (i,j) \in \mathcal{A} \tag{1c}$$

$$x_{ij} \leq y_{ij} D \qquad \forall (i,j) \in \mathcal{A}_2 \tag{1d}$$

$$h_\pi \geq 0 \qquad \forall \pi \in \Pi_{rs}, (r,s) \in \mathcal{W} \tag{1e}$$

The leader represents a network manager that aims to minimize the total system travel time (TSTT) defined as the sum of $x_{ij} t_{ij}(x_{ij})$ over all links $(i,j) \in \mathcal{A}$ subject to a budget B. It is well-known that the TAP admits a unique link flow solution and we denote $\mathsf{TAP}(\boldsymbol{y})$ this optimal solution of the parameterized TAP (1). The DNDP can be formulated as the following bilevel optimization problem :

$$\min_{\boldsymbol{y}} \quad \sum_{(i,j)\in\mathcal{A}} x_{ij} t_{ij}(x_{ij}) \tag{2a}$$

$$\text{s.t.} \quad \sum_{(i,j)\in\mathcal{A}_2} y_{ij} g_{ij} \leq B \tag{2b}$$

$$y_{ij} \in \{0,1\} \qquad \forall (i,j) \in \mathcal{A}_2 \tag{2c}$$

$$\boldsymbol{x} \in \mathsf{TAP}(\boldsymbol{y}) \tag{2d}$$

We propose to solve the bilevel optimization problem (2) by considering its high point relaxation and reformulating it using Lagrangian relaxation. Relaxing and penalizing the linking constraints (1d) leads to a separable, system optimum (SO) formulation of the DNDP that is solved by iteratively combining the solutions of parameterized SO-TAPs and binary knapsack problems. This procedure can be embedded in exact algorithms for discrete-continuous bilevel optimization problems. We discuss algorithmic considerations and present numerical results on benchmark problem instances for the DNDP.

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