An integer linear programming formulation for the maximum flow blocker problem

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1 Introduction

Let $G = (V, A)$ be a directed graph where A is a set of m arcs and V is a set of n vertices with a source $s \in V$ and a destination $t \in V$. Each arc $a \in A$ is given a *capacity* $c_a \in \mathbb{Z}_+$. We also denote an arc a as (u, v) where $u \in V$ is the tail and $v \in V$ is the head. Let y_a be a non-negative continuous variable representing the value of the flow passing through an arc $a \in A$. A *flow* from the source to the destination has to respect the following two types of constraints. The first m constraints are called the *capacity constraints* for the arcs:

$$0 \leq y_a \leq c_a, \qquad a \in A, \tag{1}$$

and the second set of n constraints are called the *flow-conservation constraints* for the vertices:

$$\sum_{a \in \delta^+(u)} y_a = \sum_{a \in \delta^-(u)} y_a, \qquad u \in V, \tag{2}$$

where $\delta^+(u) = \{(u, v) \in A : v \in V \setminus \{u\}\}$ (the outgoing arcs) and $\delta^-(u) = \{(v, u) \in A : v \in V \setminus \{u\}\}$ (the incoming arcs). Without loss of generality, we consider graphs in which the source has only one entering arc from the destination, i.e., $\delta^-(s) = (t, s)$ and the destination has only one outgoing arc, i.e., $\delta^+(t) = (t, s)$. Finally, let $\varphi(G)$ be the function returning the value of the flow outgoing from the source in a graph G:

$$\varphi(G) = \sum_{a \in \delta^+(s)} y_a, \tag{3}$$

which is also equal to the value of the flow entering the destination as well as the flow on the arc (t, s), due to the flow conservation constraints (2).

The *maximum flow problem* (MFP) asks to determine the maximum flow of a graph. It is one of the fundamental optimization problems largely studied in the literature, see e.g., [3]. A well-known min-max relation links the MFP to the *minimum cut problem* (MCP). This relationship establishes an equivalence between the two problems. Given a subset of vertices $U \subseteq V$ such that $s \in U$ and $t \in V \setminus U$, a *cut* of G, denoted by $\delta(U)$, is the subset of arcs having the head in U and the tail in $V \setminus U$. We call the value of the cut the sum of the capacities of its arcs. Accordingly, the MCP asks for finding a cut with a minimum value.

This paper studies the *blocker variant* of the *maximum flow problem* in which each arc $a \in A$ is also given an *blocker cost* $r_a \in \mathbb{Z}_+$. This problem, called the *maximum flow blocker problem* (MFBP), consists in finding a minimum-cost subset of arcs to be blocked, i.e., removed from the graph, in such a way that the maximum flow value between s and t in the remaining graph is no larger than a given threshold. The threshold, denoted by Φ, is called the *target flow*.

Applications The problem is motivated by applications in several domains. In telecommunication networks, the flow routed on an arc represents the amount of data sent from the source to the destination and passing through that arc. However, some arcs may be malfunctioning due to anomalies, failures or packet loss caused by congestion. In this context, MFBP optimal solutions allow analyzing the network resilience in case of malfunctioning arcs, modeled by removing the arcs from the graph. Precisely, considering a network with blocker costs equal to one, it is said to be resilient to f simultaneous failures if after the removal of any set of at most f arcs, there exists a flow of value larger than the target flow Φ. Accordingly, an optimal MFBP solution is a set of arcs of minimum cardinality $\zeta(G)$ such that the network is not resilient to $\zeta(G)$ simultaneous arc failures with respect to a target flow Φ. By considering $f = \zeta(G) - 1$, since $\zeta(G)$ is the minimum size of the set of arcs, removing any set of arcs of size at most $\zeta(G) - 1$ (maximum simultaneous malfunctioning arcs) ensures that the maximum flow value in the remaining graph is larger than Φ. A further application of the MFBP is in the context of monitoring civil infrastructure, where the aim is to monitor the transportation links in order to reduce illegal traffics .

Related work To the best of our knowledge, there are no articles dealing with the MFBP. However, a closely related problem, called the *maximum flow interdiction problem* (MFIP), has been largely studied in the literature. Given a graph, together with capacities and interdiction costs on the arcs and an *interdiction budget* denoted by Ψ, the MFIP consists in finding a subset of arcs of total interdiction cost no larger than Ψ to be removed from the graph in such a way that the maximum flow value is minimized. It is worth noticing that the MFIP and the MFBP have the same decision problem. Since the MFIP is $\mathcal{NP}$-hard, as demonstrated in [4], the following result holds for the MFBP.

Proposition 1 *The MFBP is* $\mathcal{NP}$*-hard.*

Our contribution The aim of this paper is to propose the first Integer Linear Programming (ILP) formulation to solve the MFBP to proven optimality. After having introduced in Section 2 a bilevel formulation for the MFBP, we derive in Section 3 an ILP formulation that features a polynomial number of variables and constraints. Then, by exploiting the structure of MFBP solutions, we establish in Section 4 a strong relationship between the MFBP and the MFIP. More precisely, we prove that any algorithm designed for one problem can be effectively used to solve the second problem. To the best of our knowledge, this is the first time a structural link between solutions of a blocker and an interdiction applied to an other optimization problem has been clarified. Finally, Section 5 reports some experimental results obtained by solving the ILP formulation proposed for the MFBP. Thanks to this set of tests, we manage to identify the features of the MFBP instances which can be solved to proven optimality.

2 A bilevel formulation for the MFBP

In this section we introduce an MFBP model which belongs to the class of blocker models, a special class of bilevel optimization formulations, see e.g., [2].

There are two types of variables in blocker models, the first-level variables associated with the so-called *leader problem* which affect the second-level variables associated with the so-called *follower problem*. For the MFBP, the leader determines a set of blocked arcs to be removed from the graph and the follower determines the maximum flow in the remaining graph.

Let us introduce a vector $\mathbf{x} \in \{0,1\}^m$ of m binary first-level variables, each variable x_a is associated to an arc $a \in A$ and it takes value 1 if and only if the arc a is blocked, i.e., removed from the graph G. Let us introduce a second vector $\mathbf{y} \in \mathbb{Q}_+^m$ of m non-negative second-level variables, each variable $y_a \geq 0$ is associated to an arc $a \in A$ and it represents the value of the flow on the arc. Therefore, a bilevel model for the MFBP reads as follows:

$$\zeta(\text{MFBP}) = \min_{\mathbf{x} \in \{0,1\}^m} \sum_{a \in A} r_a \, x_a \tag{4a}$$

$$\vartheta(\mathbf{x}) \leq \Phi, \tag{4b}$$

$$\text{where} \quad \vartheta(\mathbf{x}) = \max_{\mathbf{y} \in \mathbb{Q}_+^m} \sum_{a \in \delta^+(s)} y_a \tag{4c}$$

$$\sum_{a \in \delta^+(u)} y_a - \sum_{a \in \delta^-(u)} y_a = 0, \qquad \forall\, u \in V, \tag{4d}$$

$$y_a \leq c_a\,(1 - x_a), \qquad \forall\, a \in A. \tag{4e}$$

Constraints (4d) are the flow conservation constraints, see (2), of the vertices. Constraints (4e) model the capacity constraints of the arcs, see (1), and they impose, at the same time, a flow of value 0 on blocked arcs.

A binary realisation $\mathbf{x} \in \{0,1\}^m$ of the first-level variables is called a *blocker policy* and it generates a *non-blocked graph* $\mathcal{G}_{NB}(\mathbf{x}) = (V, \mathcal{A}_{NB}(\mathbf{x}))$, i.e., the graph with the same vertex set of G and only the non-blocked arcs $a \in A$ with $x_a = 0$ (denoted $\mathcal{A}_{NB}(\mathbf{x})$). It is worth noticing that $\vartheta(\mathbf{x})$ corresponds to the maximum value of the function $\varphi(\mathcal{G}_{NB}(\mathbf{x}))$, see equation (3), i.e., the maximum flow value in the non-blocked graph $\mathcal{G}_{NB}(\mathbf{x})$.

In Theorem 2 of [1], an important structural property of the follower problem is established. Using this property, given a blocker policy $\mathbf{x}$, the follower problem can be restated as follows:

$$\vartheta(\mathbf{x}) = \max_{\mathbf{y} \in \mathbb{Q}_+^m} \Big\{ \sum_{a \in \delta^+(s)} y_a - \sum_{a \in A} x_a\, y_a : (1), (2). \Big\}. \tag{5}$$

In this LP reformulation, constraints (4e) are replaced with the "standard" capacity constraints (1) for the arcs. The constraints of the follower do not depend anymore on the first-level variables and a penalization term is added to the new objective function to ensure that $\vartheta(\mathbf{x})$ is the maximum flow value in the non-blocked graph $\mathcal{G}_{NB}(\mathbf{x})$.

In the next section, we derive from Model (4), a single-level ILP formulation exploiting the nature of the value function $\vartheta(\mathbf{x})$, as stated in Model (5).

3 A single level ILP formulation for the MFBP

Given the vector of binary variables $\mathbf{x} \in \{0,1\}^m$ representing the blocked arcs, the objective function of (5) can be re-written as $y_{(t,s)} - \sum_{a \in A} x_a\, y_a$. This is due to the fact that the flow on the arc (t,s) is equal to the outgoing flow from the source s. Accordingly, the dual of Model (5) reads as follows:

$$\vartheta(\mathbf{x}) = \min_{\boldsymbol{\gamma} \in \mathbb{Q}_+^n, \boldsymbol{\mu} \in \mathbb{Q}_+^m} \Big\{ \sum_{a \in A} c_a\, \mu_a \ : \ \mu_{uv} + \gamma_v - \gamma_u \geq -x_{uv},\ \forall\,(u,v) \in A,\ \gamma_s - \gamma_t \geq 1 \Big\}. \tag{6}$$

This dual LP model features a vector $\boldsymbol{\mu} \in \mathbb{Q}_+^m$ of m dual non-negative variables associated with the capacity constraints (1) and a vector $\boldsymbol{\gamma} \in \mathbb{Q}_+^n$ of n dual non-negative variables associated with the flow conservation constraints (2). The dual Model (6) has a totally unimodular system of constraints and accordingly, all variables $\boldsymbol{\mu}$ take binary values in any optimal dual solutions and there exists an optimal dual solution where all variables $\boldsymbol{\gamma}$ take binary values.

By using in (4) the value function $\vartheta(\mathbf{x})$ as defined in (6), we can replace $\vartheta(\mathbf{x})$ by the objective function value in (6) since by doing so, we impose that there exists at least a solution of the follower problem ((4c) - (4e)) whose value is smaller than Φ and accordingly, its optimal solution value is also smaller than Φ. Therefore, we obtain a compact ILP formulation for the MFBP, which reads as follows:

$$
\begin{aligned}
\zeta(\text{MFBP}) \; = \; \min_{\mathbf{x},\boldsymbol{\mu}\,\in\,\{0,1\}^m,\boldsymbol{\gamma}\,\in\,\{0,1\}^n} \quad & \sum_{a\in A} r_a\; x_a && \text{(7a)}\\
& \sum_{a\in A} c_a\; \mu_a \le \Phi, && \text{(7b)}\\
\mu_{uv} + x_{uv} + \gamma_v - \gamma_u \ge 0, \quad & \forall\, (u,v) \in A, && \text{(7c)}\\
& \gamma_s - \gamma_t \ge 1. && \text{(7d)}
\end{aligned}
$$

It is worth noticing that in any optimal solution, for a given arc $a \in A$, we can have either $\mu_a = 1$ or $x_a = 1$ but not both. This is due to the fact that $r_a > 0, \forall a \in A$. If $x_a = \mu_a = 1$, then the solution obtained by setting $\mu_a = 0$ and keeping other values unchanged is still feasible and does not increase the objective function value. For this reason and due to the nature of constraints (7c) and (7d), an optimal solution of (7) is a cut in the graph G, denoted by $\delta(U^G(\mathbf{x}))$, which depends on an optimal blocker policy $\mathbf{x}$. This cut $\delta(U^G(\mathbf{x}))$ is given by the arcs $a \in A$ where $\mu_a = 1$ or $x_a = 1$ and it is the union of the set of non-blocked arcs such that $x_a = 0$ and $\mu_a = 1$ and the set of blocked arcs such that $x_a = 1$ and $\mu_a = 0$. If a variable γ_u is equal to 1, it indicates that vertex u is in the subset $U^G(\mathbf{x})$ containing the source s and if it is equal to 0, it indicates that vertex u is in the subset $V \setminus U^G(\mathbf{x})$ containing the destination t. In addition, any optimal solution $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\gamma})$ of Model (7) contains the minimum cut $\delta(U^{\mathcal{G}_{NB}}(\mathbf{x}))$ in the non-blocked graph $\mathcal{G}_{NB}(\mathbf{x})$ which is given by the arcs $a \in \mathcal{A}_{NB}(\mathbf{x})$ such that $\mu_a = 1$ and $U^{\mathcal{G}_{NB}}(\mathbf{x})$ is a set of vertices containing the source s. This is due to the fact that constraints (7c) of Formulation (7) can be equivalently rewritten as follows:

$$\mu_{uv} + \gamma_v - \gamma_u \ge 0, \quad \forall\, (u,v) \in \mathcal{A}_{NB}(\mathbf{x}), \tag{8}$$

where $\mathcal{A}_{NB}(\mathbf{x})$ are the arcs of $\mathcal{G}_{NB}(\mathbf{x})$. Together with constraints (7d), they are the standard MCP constraints for the non-blocked graph $\mathcal{G}_{NB}(\mathbf{x})$.

This ILP formulation (7), denoted by c-ILP, is called *compact formulation*, since it features a polynomial number of variables and constraints. Accordingly, it can be directly solved using an ILP solver. It is worth noticing that the structure of c-ILP is similar to the one of the formulation introduced in [4] for the MFIP. Accordingly, the next section proves that there exists a structural link between solutions of the MFIP and solutions of the MFBP.

4 Solving the MFBP via the MFIP

Let $\mathbf{w} \in \{0,1\}^m$ be a vector of binary variables associated with the set of arcs A of a graph G, each variable w_a takes value 1 if and only if the arc a is removed from the graph G, i.e., *interdicted*. A binary realization of variables $\mathbf{w} \in \{0,1\}^m$ is called an *interdiction policy* and it generates a *non-interdicted graph* $\mathcal{G}_{NI}(\mathbf{w}) = (V, \mathcal{A}_{NI}(\mathbf{w}))$, i.e., the graph induced by the set of non-interdicted arcs $a \in A$ with $w_a = 0$ (denoted $\mathcal{A}_{NI}(\mathbf{w})$).

By using two sets of binary variables $\boldsymbol{\beta} \in \{0,1\}^m$ and $\boldsymbol{\alpha} \in \{0,1\}^n$, the compact ILP formulation for the MFIP, of [4], is shown in Model (9).

$$
\begin{aligned}
\zeta(\text{MFIP}) \; = \; \min_{\mathbf{w},\boldsymbol{\beta}\,\in\,\{0,1\}^m,\boldsymbol{\alpha}\,\in\,\{0,1\}^n} \quad & \sum_{a\in A} c_a\; \beta_a && \text{(9a)}\\
& \sum_{a\in A} q_a\; w_a \le \Psi, && \text{(9b)}\\
\beta_{uv} + w_{uv} + \alpha_v - \alpha_u \ge 0, \quad & \forall\, (u,v) \in A, && \text{(9c)}\\
& \alpha_s - \alpha_t \ge 1. && \text{(9d)}
\end{aligned}
$$

It is worth noticing that constraints (9c) and (9d) share the same structural form as constraints (7c) and (7d), respectively. For this reason, an optimal solution of (9) is also a cut in the graph G, denoted by $\delta(U^G(\mathbf{w}))$, which depends on an optimal interdiction policy $\mathbf{w}$. More precisely, the cut $\delta(U^G(\mathbf{w}))$ is given by the arcs $a \in A$ where $\beta_a = 1$ or $w_a = 1$ and it is the union of the set of non-interdicted arcs such that $\beta_a = 1$ and $x_a = 0$ and the set of interdicted arcs such that $w_a = 1$ and $\beta_a = 0$. Constraint (9b) imposes that the total interdiction cost, i.e., the interdiction cost induced by the set of interdicted arcs, does not exceed the interdiction budget Ψ. The objective function (9a) minimizes the sum of the capacities of the non-interdicted arcs contained in the cut $\delta(U^G(\mathbf{w}))$. Moreover, the minimum cut in the non-interdicted graph is given by the arcs $a \in \mathcal{A}_{NI}(\mathbf{w})$ such that $\beta_a = 1$.

The next proposition shows how to obtain an optimal MFBP solution starting from an optimal MFIP solution of an instance where the interdiction costs are set to the arc-capacities, the arc-capacities are set to the blocker costs and the interdiction budget is set to the target flow value.

Proposition 2 *Given an instance of the MFBP* $(G, \mathbf{c}, \mathbf{r}, \Phi)$*, an optimal solution is given by an optimal solution of the MCP solved in the non-interdicted graph* $\mathcal{G}_{NI}(\mathbf{w})$ *associated to an optimal solution* $\mathbf{w}$ *of the MFIP instance* $(G, \mathbf{q}, \mathbf{c}, \Psi)$*.*

Sketch of proof : Let $(G, \mathbf{c}, \mathbf{r}, \Phi)$ be an instance of the MFBP. We set $r_a = c_a$ and $c_a = q_a$ for every arc $a \in A$ and $\Phi = \Psi$. Once the MFIP (see Model ((9a)-(9b))) is solved, its optimal solution $(\mathbf{w}, \boldsymbol{\beta}, \boldsymbol{\alpha})$ corresponds to an optimal MFBP solution. Indeed, since $\mathbf{w}, \boldsymbol{\beta} \in \{0,1\}^m$ and $\boldsymbol{\alpha} \in \{0,1\}^n$, the constraints of the MFIP, i.e, constraints (9b), (9c) and (9d) become identical to the constraints of the MFBP, i.e, Constraints (7b), (7c) and (7d), respectively. The variable values $\boldsymbol{\beta}$ corresponds to the blocked arcs, while the variable values $\mathbf{w}$ corresponds to the arcs remaining in the cut $\delta(U^G(\boldsymbol{\beta}))$. According to the objective function (9a) of the MFIP, the total cost of the blocked arcs will be minimized, leading to the optimal solution value of the MFBP. Moreover, when solving a MFIP, the variable values $\boldsymbol{\beta}$ corresponds to the arcs of the minimum cut in the non-interdicted graph $\mathcal{G}_{NI}(\mathbf{w})$, as explained previously. Accordingly, the blocked arcs corresponds to the minimum cut in the non-interdicted graph $\mathcal{G}_{NI}(\mathbf{w})$.
The same reasoning can be used to prove that given an instance of a MFBP, an optimal solution of a MFIP can be found by solving a MCP in the non-blocked graph.

$\square$

Using Proposition 2, the next proposition provides additional complexity results for the MFIP and the MFBP.

Proposition 3 *The MFIP and the MFBP are strongly* $\mathcal{NP}$*-hard, even if all arcs have the same capacity.*

Sketch of proof : Let $\hat{\mathbf{q}} \in \mathbb{Z}_+^m$ be an interdiction cost vector where all values are identical. Starting from an MFIP instance $(G, \mathbf{c}, \hat{\mathbf{q}}, \Psi)$, i.e, a MFIP instance where all arcs have the same interdiction cost, we set $\Phi = \Psi$, $r_a = c_a$ and $c_a = \hat{q}_a$ for all arcs $a \in A$. Once this MFIP is solved, the minimum cut remaining in the non-interdicted graph corresponds to an optimal solution of a MFBP instance $(G, \hat{\mathbf{q}}, \mathbf{c}, \Psi)$, i.e, a MFBP instance where all arcs have the same capacity. In [4], the MFIP has been shown to be strongly $\mathcal{NP}$-hard even if all arcs have the same interdiction cost. Accordingly, the MFBP is strongly $\mathcal{NP}$-hard, even if all arcs have the same capacity. Moreover, as the MFIP and the MFBP share the same decision problem, the MFIP is also strongly $\mathcal{NP}$-hard, even if all arcs have the same capacity.

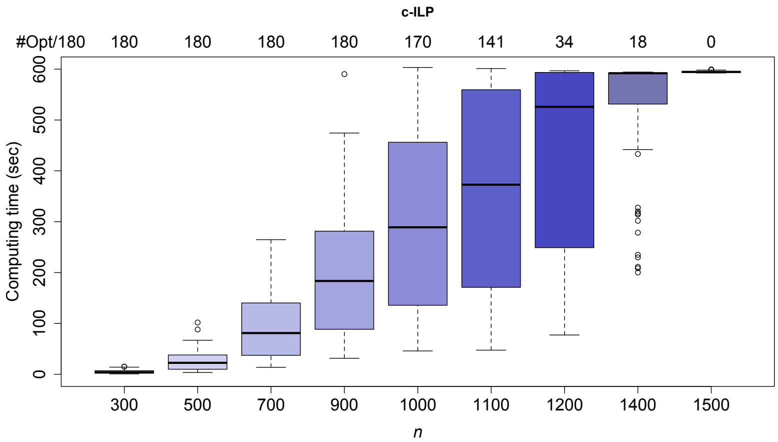
$\square$

5 Experimental results

In this section, we test the efficiency of the compact formulation. More precisely, we are seeking to achieve the limits of Model (7). By doing so, we can determine the maximum size of instances that can be solved to proven optimality within a fixed amount of CPU time limit of

600 seconds. For this study, we use CPLEX 12.7.0. We consider graphs, generated randomly, with different sizes defined by the number of vertices n. For every instance, we consider three different target flow values defined as a percentage of the maximum flow value ($30\%, 60\%$ and 90%). In Figure 1, we present the computing time boxplot of c-ILP. We show in this graph, the time spent by the formulation through the quartiles. The lines extending vertically indicate the variability outside the upper and lower quartiles. The y-axis is the computing time and the x-axis represents values of n. On the top part of the figure, we report for each value of n, the total number of instances solved to proven optimality (#opt) out of the total number of instances considered (180). This boxplot shows that the formulation manages to solve to proven optimality graphs with up to 900 vertices within the time limit. For graphs with up to 700 vertices, c-ILP reaches an optimal solution in a very reasonable time (less than 300 seconds for the worst case). However, it starts to face some difficulties when dealing with graphs of 1000 vertices, where ten instances remain unsolved. Subsequently, the number of instances solved to proven optimality decreases to 18 out of 180 for $n = 1400$ and for larger graphs (1500 vertices), no instances have been solved to proven optimality within the time limit.

FIG. 1: Computing time boxplot of c-ILP on `SYNTHETIC` instances



6 Conclusions et perspectives

In this paper, we have studied the maximum flow blocker problem. By exploiting the nature of the problem, we derive the first compact formulation to solve it to proven optimality. We then demonstrate a structural link between solutions of the MFBP and solutions of the MFIP. This relationship allows us to obtain an optimal solution of one problem given an optimal solution of the other. As a future line of research, one may actually wonder whether it is possible to extend this result to other general network flow blocker problems.

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