Parameterized complexity: a two-dimensional approach to study scheduling problems

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Complexity theory is a general framework which aims to classify computer science problems depending on the time and/or space it takes to solve them. In the context of scheduling it has been successfully used to distinguish between the polynomial-time solvable problems and the NP-complete ones [7]. However the vast majority of scheduling problems, even seemingly basic ones such as $1|r_j|L_{max}$, has been proved strongly NP-hard [10] over the years. Upon reaching NP-hardness this quickly, it becomes difficult to establish anything deeper about scheduling problems within the scope of classical complexity theory.

To answer this, parameterized complexity theory gives additional tools for a refined analysis of such hard scheduling problems. Given a parameter k and denoting n the input size, a problem is called $fixed-parameter\ tractable$ (FPT) parameterized by k if it can be solved in time $\mathcal{O}(poly(n) \times f(k))$ with f an arbitrary computable function [4]. The idea is to identify k as the limiting property and give a polynomial time algorithm for all instances with a bounded value of k. When the studied problem is believed to not be FPT, many complexity classes [5] are available as parameterized analogues to NP. Here we use class para-NP which was defined in [6]. By proving para-NP-hardness one shows that the problem remains NP-hard for some fixed value of the parameter. This rules out any chance for this problem to be FPT - unless $P = NP$.

While parameterized complexity theory has been successfully applied to a lot of computer science areas, it has started to be studied extensively on scheduling only quite recently. One of the few older results was from [2] and showed that $P|prec, p_j = 1|C_{max}$ is W[2]-hard parameterized by the number m of machines (see [5] for a definition of the W-hierarchy). Some interesting and recently studied parameters include the width w of the partial order representing the precedence constraints [3], pathwidth μ which is the maximum number of overlapping job time windows at any given time [1, 8, 9, 12] and maximum delay value ℓ_{max} when delays are added to precedence constraints [12]. On top of providing new exact methods via FPT results, parameterized complexity helps to further classify scheduling problems. On top of the P vs NP-hard frontier, each parameter gives an additional FPT vs para-NP-hard frontier on the strongly NP-hard scheduling problems.

In this contribution we apply this framework to scheduling problems with time windows and precedence delays. We give several parameterized frontiers with pathwidth μ and maximum delay value ℓ_{max} and show what it means about the corresponding scheduling problems. In particular we recall that problems $1|prec, r_j, \bar{d}_j|C_{max}$ and $P|prec, p_j = 1, r_j, \bar{d}_j|C_{max}$ are FPT parameterized by pathwidth μ [12, 13], which illustrates how powerful pathwidth μ is when dealing with arbitrary processing times and/or precedence constraints. However when delays are added to precedence constraints, we show that parameter μ becomes inefficient even when restricting ourselves to the single machine scheduling of coupled tasks with unit processing times and equal precedence delays. This is done by proving that $1|(1, \ell, 1), r_j, \bar{d}_j|C_{max}$ is para-NP-hard parameterized by μ. This is an improvement over a result which was presented at ROADEF 2022 where there were chains with an arbitrary number of available delay values [11]. Finally we show that such problems become FPT when parameters μ and ℓ_{max} are combined [12].

This exemplifies the two-dimensional approach offered by parameterized complexity, where the strength of the parameter can be played with on top of the expressiveness of the scheduling problem. We believe that such an approach could be successfully used on other scheduling problem families.

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