A quantum pricing-based column generation framework for hard combinatorial problems

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1 Introduction

Combinatorial optimization is at the heart of many real-world problems. It consists in finding the "best" out of a finite, but prohibitively large, set of options. Column generation is an iterative method that was developed to solve this kind of difficult mathematical problems, such as linear formulations where the problem may be too large to consider all options explicitly. In this work, we present a complete hybrid classical-quantum algorithm involving a quantum sampler based on neutral atom platforms. This approach is inspired by classical column generation frameworks developed in the field of Operations Research and shows how quantum procedures can assist classical solvers in addressing hard combinatorial problems. We benchmark our method on the Minimum Vertex Coloring problem and show that the proposed hybrid quantum-classical column generation algorithm can yield good solutions in relatively few iterations. This presentations is based on previous our work [1].

2 Methodology

As previously discussed, solving the pricing sub-problems is usually the bottleneck in column generation-based algorithms since it comes to solving different instances of a hard combinatorial problem multiple times. To overcome this problem, we propose a quantum pricing algorithm that can find the (near-) optimal solution faster than the classical one (*i.e.*, where no QPU is involved). For this purpose, let us now describe the column generation-based framework proposed to solve the Minimum Vertex Coloring problem.

First, a minimal sub-set $S' \subseteq S$ of independent sets is generated in such a way that it ensures a feasible solution for the extended formulation foe-cei. As previously discussed, the most trivial way to build the initial set S' of independent sets is generating only the singletons in the graph; this simple approach always provides a solution for the RMP.

The classical part of the proposed hybrid approach is related to the Restricted Master. Once the initial set S' is created, the RMP is built and then solved on its linear relaxation form (see formulation rfoe-rce) by a classical solver (*e.g.*, GPLK). The values of the dual variables are also given by the classical solver by running a built-routine after solving each version of the RMP (i.e., with different sub-sets of variables).

The next steps are related to the pricing sub-problems, in which the PSP is solved by applying the values of the related dual variables from the solved RMP. As previously discussed, this step comes to finding independent sets whose weight is strictly greater than 1. If such elements exist, then they are added to S'. As we detail in the following, we propose a quantum sampler that is specifically tailored to output multiple independent under the aforementioned conditions For each new independent set found by solving the related pricing sub-problem, a new variable is created and added to the sub-set S'. Then, the RMP is solved again with the new columns (*i.e.*, independent sets converted into variables). These last steps are repeated until no column is generated by the PSP. Finally, the final RMP is solved with all generated variables (*i.e.* independent sets) with the integrality constraints cei, as previously discussed.

3 Numerical Results

We compared the number of iterations needed to be run on different graph classes, orders, and densities by applying different approaches: Classical Column Generation (CG), Greedy CG, SA CG, Noiseless Quantum CG, Classical Greedy, and Quantum Greedy. We applied the AIPR-HDR strategy for redesigning each pricing sub-problem within the Quantum CG framework. While this indicator refers to how many times the PSP was solved within both classical and quantum column generation frameworks for coloring a given graph, it indicates how many independent sets were generated during the while-loop on proposed algorithm by using both classical and quantum methods as previously discussed. First, we observe that the Quantum Greedy approach has the same overall performance as its classical counterpart, showing that our quantum sampler can solve the Maximum Independent Set problem efficiently. Also, both strategies have the same linear behavior related to the size of the graph, *i.e.*, the number of edges it contains, being most impacted by dense graphs. This behavior is expected since the size of each independent set gets smaller as the set of edges gets larger. Hence, more iterations have, in general, to be done to cover all vertices of a dense graph. Also, while outperforming the Classical CG approach on almost every graph class (in terms of the number of iterations), both Classical and Quantum Greedy algorithms had their performance slightly decreased on UD graphs. Finally, taking advantage of the related superposition aspect, the proposed Quantum CG outperformed all other approaches on all graph classes. For instance, while the Quantum CG algorithm needed less than 4 (resp. 6) sampling iterations for all sparse and dense (resp. 0.5-density) non-UD graphs, Quantum and Classical Greedy approaches (resp. Classical CG algorithm) needed up to 10 (resp. 12) pricing interactions to solve the same graph class.

First, we observed that our proposed Quantum CG approach has the best overall performance. Indeed, it could find the optimal solution in almost all instances; our approach could not find the best solution only for some 13-vertex non-UD graphs. Also, unlike all other approaches, the Quantum CG is not impacted by the graph class; while the Classical CG could better perform on dense graphs, both Classical and Quantum Greedy approaches are more stable on UD graphs. Also, the proposed Quantum CG algorithm could reduce the average gap on 12-vertex non-UD (resp. 13-vertex UD) graphs from roughly 19% (resp. 11%) to 0% when compared to the Quantum Greedy (resp. Classical CG) approach.

4 Concluding remarks

Our proposed quantum pricing-based approach also outperformed both stochastic classical approaches in most of the instances, especially those related to UD and sparse graphs. Even though the quality of the solutions remains the same. The Noiseless Quantum CG could reduce by 50% the number of iterations on sparse graphs when compared to SA-based pricing. Even though the Greedy CG has fewer pricing iterations on some graph classes, as in bigger non-UD graphs with 20% and 50% of density the average gap could be reduced by 80% when our proposed Quantum CG was applied on the same graph classes. This indicates that random sampling to find independent sets cannot solve pricing sub-problems effectively.

References

[1] da Silva Coelho, W., Henriet, L., Henry, L. P. (2023). Quantum pricing-based column-generation framework for hard combinatorial problems. Physical Review A, 107(3), 032426.