Introduction to neighborhood dominance inequalities

Pierre Fouilhoux, Lucas Létocart, Yue Zhang

LIPN, Université Sorbonne Paris Nord, CNRS UMR 7030, Villetaneuse, France

{pierre.fouilhoux, lucas.letocart, yue.zhang}@lipn.univ-paris13.fr

Mots-clés : *integer linear programming, polyhedral approach, neighborhood dominance inequalities, bi-objective integer linear optimization.*

1 Introduction

The polyhedral approach is a prevalent methodology for strengthening (mixed) integer linear programs ($\max f^{obj}(x)$ s.t. $x \in \mathcal{X} = \{Ax \leqslant b,\ x \in \mathbb{N}^n\}$). The main idea is to provide valid inequalities that cut off the fractional points of the polyhedron $\tilde{\mathcal{X}} = \{Ax \leqslant b,\ x \in \mathbb{R}^n\}$ and remain valid for the convex hull polytope $conv(\mathcal{X})$ of all integer feasible points. As these definition does not involve the objective function, such valid inequalities often cut fractional points far away from the optimal point.

Consequently, a very natural question arises: why not discard points of no interest w.r.t. f^{obj}*-cut* inside $conv(\mathcal{X})$ as well? We introduce below two categories of inequalities different from the valid inequalities.

Definition 1 *An inequality is called f^{obj}-valid (resp. f^{obj}-cut) if it preserves the integer optimum (resp. preserves the integer optimum and cuts at least one optimal point of $\tilde{\mathcal{X}}$).*

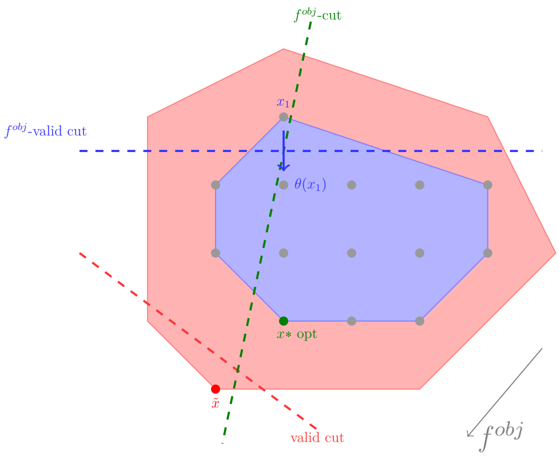

FIG. 1: An illustration of different inequalities in the variables domain, where the relaxed polyhedron is drawn in red, the blue polygon represents the convex hull of all integer points, and feasible points are the black points. The valid inequality in red violates the LP optimum $\tilde{x}$, the f^{obj}-valid inequality in blue cuts off x_1 dominated by $\theta(x_1)$, and the f^{obj}-cut in green violates both dominated integer point x_1 and LP optimal $\tilde{x}$.

Anne-Elisabeth Falq [1] proposed, for the first time, the *neighborhood-based dominance inequalities* about the initial idea of eliminating *locally dominated feasible solutions*, which are f^{obj}-cut and in some cases, improve the relaxation bounds and significantly reduce the polytope by cutting some locally dominated points, according to the following definition.

Definition 2 *Given two points $x \neq x'$ in $\mathcal{X}$, x is dominated by x' if $f^{obj}(x) > f^{obj}(x')$.*

With a fixed *neighborhood relation* θ (as considered for iterative methods in meta-heuristics), the idea is, for each feasible solution $x \in \mathcal{X}$, to compare it with its neighbor $\theta(x)$ and cut off the dominated one. The final θ-dominance inequalities (1) require a clear definition of the following elementary functions:

Definition 3 *$\theta : \mathbb{R}^n \to \mathbb{R}^n$ denotes an operation on a subset of feasible solutions $\mathcal{X}' \subseteq \mathcal{X}$.*

Definition 4 *Let Π be a linear identifying function : $\mathbb{R}^n \to \mathbb{R}^+$, such that $\Pi(\mathcal{X}) \subseteq \mathbb{N}$ and $\forall x \in \mathcal{X}' \subseteq \mathcal{X}$, $\Pi(x) = 0$.*

Definition 5 *Δ a linear variation function : $\mathbb{R}^n \to \mathbb{R}$, $\Delta(x) = f^{obj}(\theta(x)) - f^{obj}(x)$, denotes the objective variation from x to $\theta(x)$.*

Proposition 1 *[1] In a maximization problem, given a solution $x \in \mathcal{X}$, x is eliminated by the θ-dominance inequality*

$$\Delta(x) \leqslant M\Pi(x) \tag{1}$$

if x is dominated by $\theta(x)$, where M is an upper bound of $\{\Delta(x) \mid x \in \mathcal{X}\}$.

In Anne-Elizabeth's thesis [1], the θ-dominance inequalities (1) were shown to be both f^{obj}-valid and f^{obj}-cut for the unrestrictive common due date problem. Whereas, for max-cut problems, [1] proved that the θ-dominance inequalities (1) do not violate any LP optimum on $\tilde{\mathcal{X}}$. Moreover, we notice that the θ-dominance inequalities cannot always be directly applied in case of the infeasibility of $\theta(x)$, for example for the insert operations of the knapsack problem. In the next section, we give a generalization of the θ-dominance inequalities.

2 θ-Dominance inequality

Operation θ feasibility In inequalities (1), solution x is cut off if it is dominated by $\theta(x)$ with an objective variation $\Delta(x) > 0$. While comparing x with an infeasible dominant $\theta(x)$ makes no sense, we complete the identifying function Π with an additional variable $\pi \in \mathbb{R}$ representing the feasibility of $\theta(x)$. Denote $Ax \leqslant b = \{a_i x \leqslant b_i, \forall i \in [1, m]\}$ and $x' = \theta(x)$. If x' is feasible then $a_i x' - b_i \leqslant 0, \forall i \in [1, m]$. Let $\pi = \begin{cases} 0, \text{ if } x' \text{ is feasible} \\ \text{the maximum violation of } a_i x' - b_i \; \forall i \in [1, m] \end{cases}$, equivalently $\pi = \max(0,\, a_1 x' - b_1, \ldots, a_m x' - b_m)$.

We first apply our generalized θ-dominance inequalities on knapsack problems and show that (1) could rarely be f^{obj}-cut. Nevertheless, the θ-dominance inequalities significantly reduce the considered polytope by deleting the locally dominated points, the big-M constraints may make the linear programs ill-conditioned. Therefore we are interested in this polytope, called the *θ-dominant polytope*.

Polyhedral aspect motivation With the PORTA software[1], we analyze several knapsack instances and study the generalized facet-defining inequalities on the θ-dominant polytope.

Extension to bi-objective integer linear optimization problems Due to the enumeration of the exponentially large Pareto optimal set and the exploration in high dimensional decision space, the bi-objective B&B algorithm is very time-consuming. In the hope that the θ-dominance inequality may improve the relaxation bounds and reduce the dominated variable space, we consider the minimum objective variation in inequalities (1) for the bi-objective case.

References

[1] Anne-Elisabeth Falq. *Dominances in linear programming: scheduling around a common due date.* PhD thesis, Sorbonne Université, 2020.

[1] https://porta.zib.de/