Adaptive Partitioning for Chance Constraint Programs with Finite Support

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1 Introduction

Chance-Constrained Stochastic Programs (CCSPs) play a crucial role in decision-making under uncertainty, offering a framework for finding optimal solutions while considering the probability of constraint violation. Originating from the seminal work of Charnes and Cooper, CCSPs have found applications in diverse domains such as power systems, vehicle routing, finance, and contextual optimization.

In this paper, we delve into the realm of CCSPs with finite support, addressing the challenge of efficiently solving these complex problems. Our focus lies in developing an adaptive partitioning method (APM) inspired by previous works, such as [2] and [3]. The proposed APM is designed to find optimal solutions to CCSPs with finite support after a finite number of iterations. The key contribution of our work lies in how the APM is extended for solving CCSPs, which is not straightforward due to the inherent combinatorial structure of chance constraints. A complete redesign of the APM components is necessary in this case.

The suggested methodology is formulated to be applicable irrespective of the specific structure of the chance constraint. In other words, it is capable of handling both single and joint chance constraints. Furthermore, if linear CCSPs are considered, the proposed method functions regardless of whether the chance constraint involves uncertainty on the right-hand side or the left-hand side.

2 Problem Statement

CCSPs involve determining the optimal value of a decision variable, subject to constraints influenced by uncertain parameters. We want to obtain the optimal value of a decision variable $x \in \mathcal{X} \subseteq \mathbb{R}^n$ that minimizes the objective function $f : \mathcal{X} \to \mathbb{R}$. This decision variable has to be contained by a set X that depends on an uncertain parameter $\xi \in \Xi$ with a probability of $(1 - \tau)$. Here, the parameter $\tau \in [0, 1]$ represents the risk tolerance of the decision-maker. The generic CCSP reads

$$
\begin{aligned}
v^* = \min_{x \in \mathcal{X}} \quad & f(x) && \text{(1a)}\\
\text{s.t.} \quad & \mathbb{P}_\xi[x \in X(\xi)] \geq 1 - \tau. && \text{(1b)}
\end{aligned}
$$

More specifically, we focus on CCSPs whose uncertain parameters $\xi \in \Xi$ have finite support. The uncertain parameters belong to the set $\Xi := \{\xi^s : s \in S\}$ where each $\xi^s \in \mathbb{R}^d$ is a multi-dimensional vector representing a single realization of the uncertain parameters with probability q_s and S is a set of scenarios. By introducing a binary variable $z_s \in \{0, 1\}$ for each $s \in S$, Model (1) can be reformulated, see, e.g., [1],

$$
\begin{aligned}
v^* = \min_{x \in \mathcal{X}} \quad & f(x) && \text{(2a)}\\
\text{s.t.} \quad & z_s = \mathbb{1}(x \in X^s), \quad s \in S, && \text{(2b)}\\
& \sum_{s \in S} q_s z_s \geq 1 - \tau, && \text{(2c)}\\
& z_s \in \{0, 1\}. \quad s \in S && \text{(2d)}
\end{aligned}
$$

In Model (2), $\mathbb{1}$ is the indicator function, and $X^s = X(\xi^s)$ is the set of feasible decisions for realization ξ^s.

3 Adaptive Partitioning Method

Our approach is based on creating partitions of the scenarios set.

Definition 1 *A partition* $P = \{p_1, p_2, \ldots, p_{|P|}\}$ *is a collection of non-empty subsets of the scenario set* S *such that* $\bigcup_{p \in P} p = S$ *and* $p_i \cup p_j = \emptyset$*, for all* $p_i, p_j \in P$.

By encapsulating scenarios in a subset $p \in P$, a smaller-size chance-constrained problems is proposed in [1]. In this reduced CCSP each subset $p \in P$ represents a unique scenario and the feasible set for $p \in P$ is $X^p = \bigcap_{s \in p} X^s$. We use this reduced CCSP as a component of the proposed APM, it reads

$$
\begin{aligned}
v^{\mathrm{L}}(P) = \min_{x \in \mathcal{X}} \quad & f(x) && \text{(3a)}\\
\text{s.t.} \quad & z_p = \mathbb{1}(x \in X^p), \quad p \in P, && \text{(3b)}\\
& \sum_{p \in P} q_p z_p \geq \sum_{p \in P} q_p - \tau, && \text{(3c)}\\
& z_p \in \{0, 1\}, \quad p \in P, && \text{(3d)}
\end{aligned}
$$

where for each subset $p \in P$ the probability $q_p = \min_{s \in p} q_s$. In Model (3) no constraint aggregation is carried out. In fact, Model (3) has as many constraints as Model (2). Nevertheless, Model (3) features $|P|$ binary variables, indicating that it requires fewer computations to solve compared to Model (2), which utilizes $|S|$ binary variables.

Proposition 1 (from [1]) *The partitioned model* (3) *is a relaxation of the CCSP* (2)*, i.e.,*

$$v^* \geq v^L(P).$$

The concept behind the APM involves an iterative adjustment of a partition, denoted as P^j. The objective is to modify P^j in a way that in each subsequent iteration of the APM, a strictly increasing lower bound value, $v^{\mathrm{L}}(P)$, is achieved. This iterative process continues until a feasible solution of the model is attained, or the calculated optimality gap meets a predefined threshold parameter, ε. The general procedure of the APM is outlined in Algorithm 1, consisting of three key steps: lower bound computation, upper bound computation, and the modification of P^j. As long as $|P^j|$ increases over the course of the iterations, Algorithm 1 terminates in a finite number of iterations and recovers the optimal solution of Model (2).

Algorithm 1: Adaptive Partitioning Method.

Input: scenario set S, stopping criterion $\varepsilon \in (0, 1)$.
Output: optimal solution x^* of Model (2).
Initialize: $j \leftarrow 0$, $v^{\mathrm{U}} \leftarrow +\infty$, $v^{\mathrm{L}} \leftarrow -\infty$.

```
Design the first partition P^0.
while (v^U − v^L)/v^U ≥ ε do
    Find x̲^j, the solution of Model (3) for the partition P^j.
    if v(x̲^j) > v^L then set x^L ← x̲^j and v^L ← v(x̲^j).
    Find x̄^j by projecting x̲^j in the feasible set of Model (3).
    if v(x̄^j) < v^U then set x^U ← x̄^j and v^U ← v(x̄^j).
    Modify P^j to obtain a new partition P^{j+1}.
    Increment iteration j ← j + 1.
end
return x^U
```

References

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