Finite adaptability in robust optimization : asymptotic optimality and tractability

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Context

Of particular importance in operations research are two-stage robust optimization problems of the form

$$\begin{array}{rl} \displaystyle\min_{\boldsymbol{x},\boldsymbol{y}(\cdot)} & \boldsymbol{c}^\top \boldsymbol{x} + \displaystyle\max_{\omega\in\Omega} \boldsymbol{d}^\top \boldsymbol{y}(\omega) \\ \text{s.t.} & A(\omega)\boldsymbol{x} + B(\omega)\boldsymbol{y}(\omega) \leqslant \boldsymbol{b}(\omega) \quad \forall \omega \in \Omega\,, \end{array} \tag{CompAdapt(Ω)}$$

where Ω is the uncertainty set, $A(\cdot)$ and $B(\cdot)$ are input matrices depending on the uncertainty, and $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are deterministic vectors. The variables are $\boldsymbol{x}$ and $\boldsymbol{y}(\cdot)$: the variable $\boldsymbol{x}$ is the "here-and-now" variable, whose value has to be determined without knowing the exact $\omega \in \Omega$ that will be selected, contrary to $\boldsymbol{y}(\cdot)$—the "wait-and-see" variable—whose value can arbitrarily depend on ω.

These problems are especially relevant in situations where recourse actions are still possible after some uncertainty has been revealed, and have been extensively studied in the literature [2]. A fundamental technique to solve this problem, *finite adaptability*, has been introduced in 2010 by Bertsimas and Caramanis in their seminal paper *Finite Adaptability in Multistage Linear Optimization* [1]. It consists in restricting the range of $\boldsymbol{y}(\cdot)$ to piecewise constant functions with at most k distinct values :

$$\begin{array}{rl} \displaystyle\min_{\substack{\Omega_1\cup\cdots\cup\Omega_k \\ \boldsymbol{x},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k}} & \boldsymbol{c}^\top \boldsymbol{x} + \displaystyle\max_{i\in[k]} \boldsymbol{d}^\top \boldsymbol{y}_i \\ \text{s.t.} & A(\omega)\boldsymbol{x} + B(\omega)\boldsymbol{y}_i \leqslant \boldsymbol{b}(\omega) \quad \forall i \in [k] \;\; \forall \omega \in \Omega_i\,, \end{array} \tag{Adapt$_k$(Ω)}$$

where the Ω_i are constrained to form a partition of Ω. A natural question is how well finite adaptability approximates *complete adaptability*, which consists in solving (CompAdapt(Ω)) in its full generality, i.e., without any restriction on the variable $\boldsymbol{y}(\cdot)$.

Bertsimas and Caramanis claimed that a certain condition is sufficient to ensure that finite adaptability is asymptotically optimal. We show that this condition is not correct and propose an alternative one. In this work we also initiate the study of tractability of finite adaptability.

Asymptotic optimality

When Ω is a polytope of $\mathbb{R}^n$ and when $A(\cdot)$ and $B(\cdot)$ depend affinely on the uncertainty, a proposition in the aforementioned paper states that, under a continuity assumption,

$$\lim_{k\to+\infty} \mathrm{val}(\mathrm{Adapt}_k(\Omega)) = \mathrm{val}(\mathrm{CompAdapt}(\Omega))\,, \tag{1}$$

where val(P) denotes the optimal value of a problem (P). This continuity assumption is

Continuity assumption : *For any* $\varepsilon > 0$*, for any* $\omega \in \Omega$*, there exists* $\delta > 0$ *and a point* $(\boldsymbol{x}, \boldsymbol{y})$*, feasible for* $A(\omega), B(\omega), b(\omega)$ *and within* ε *of optimality, such that* $\forall \omega' \in \Omega$ *with* $\|\omega - \omega'\| \leqslant \delta$*, the point* $(\boldsymbol{x}, \boldsymbol{y})$ *is also feasible for* $A(\omega'), B(\omega'), b(\omega')$*.*

A point $(\boldsymbol{x}, \boldsymbol{y})$ is *within* ε *of optimality* if $\boldsymbol{c}^\top \boldsymbol{x} + \boldsymbol{d}^\top \boldsymbol{y} - \varepsilon$ is at most $\mathrm{val}(\mathrm{CompAdapt}(\Omega))$.

Proposition 1 *There exist two-stage robust optimization problems for which the above Continuity assumption is verified, yet equality* (1) *is not.*

To fix this we propose the following continuity assumption.

Modified continuity assumption : *For any* $\varepsilon > 0$*, there exists* $\boldsymbol{x}$ *such that, for any* $\omega \in \Omega$*, there exist* $\delta > 0$ *and* $\boldsymbol{y}$ *satisfying the following two conditions simultaneously :*

- $(\boldsymbol{x}, \boldsymbol{y})$ *is feasible for* $A(\omega'), B(\omega'), b(\omega')$ *for all* $\omega' \in \Omega$ *with* $\|\omega - \omega'\| \leqslant \delta$*.*
- $(\boldsymbol{x}, \boldsymbol{y})$ *is within* ε *of optimality.*

This "modified continuity assumption" is more restrictive in the sense that every problem satisfying the "modified continuity assumption" satisfies the original "continuity assumption." It is not difficult to see that the converse is not true.

We establish the asymptotic optimality of finite adaptability under the modified assumption. This shows that the general message of Bertsimas and Caramanis asymptotic result is correct, namely that, under some continuity assumption, finite adaptability converges to complete adaptability.

Theorem 2 *Assume that* Ω *is a compact subset of* $\mathbb{R}^n$*. If the "modified continuity assumption" holds, then*

$$\lim_{k \to +\infty} \mathrm{val}(\mathrm{Adapt}_k(\Omega)) = \mathrm{val}(\mathrm{CompAdapt}(\Omega)) .$$

Tractability

Even in very specific cases $(\mathrm{Adapt}_k(\Omega))$ is NP-hard [1]. To the authors' knowledge, the question of identifying relevant special polynomial cases has not been addressed yet. The only non-immediate result we have established is the following.

Proposition 3 *Assume* Ω *is a segment of* $\mathbb{R}$*, the matrices* $A(\cdot)$ *and* $B(\cdot)$ *are constant, and* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are continuous variables. Then the feasibility version of* $(\mathrm{Adapt}_k(\Omega))$ *can be solved in polynomial time.*

A key result to establish this proposition is the following lemma.

Lemma 4 *Under the condition of the proposition, we can assume without loss of generality that each* Ω_i *is a bounded interval.*

A counter-part of this lemma for Ω of dimension larger than one would be the existence of a partition into convex Ω_i's. Unfortunately, such a result is not correct even for $k = 2$. Extending Proposition 3 with uncertainty set Ω of higher dimension seems therefore to be very challenging.

Références

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