An improved variant of the Iterated Inside Out algorithm for solving the optimal transport DOTmark Instances

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The optimal transport (OT) is one the most paradigmatic among network optimization problems. Solving to optimality OT implies finding the minimum cost transportation plan that moves quantities of a single item from a set of sources M to a set of destinations N. A connection among any pair of a source i and a destination j exists and is characterized by a unitary transportation cost c_{ij}. Given respectively a_i and b_j the item quantities available at the sources and requested by destinations, the problem can be formulated by means of the following LP model [9].

$$\min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} \tag{1}$$

$$\sum_{j \in N} x_{ij} = a_i \ \forall i \in M \tag{2}$$

$$\sum_{i \in M} x_{ij} = b_j \ \forall j \in N \tag{3}$$

$$x_{ij} \geq 0 \ \forall i \in M, j \in N. \tag{4}$$

Since [9], a large literature florished on OT and primal network simplex (NS) algorithms, *e.g.*, [8], for the minimum cost flow problem (that generalizes OT) were denoted as the best performing approaches [6] for OT (performing also much better than LP-based algorithms). The interest in the OT has been renewed recently because of machine learning and artificial intelligence applications requiring the computation of the distance between pairs of images or probability distributions with source and destination quantities being pixel intensities or probability measures respectively. A set of image processing instances, called DOTmark, is nowadays the benchmark for OT and the NS of [7] was indicated in [3] as the best performing algorithm for those instances. Interior point based approaches, still less efficient than [7], were also proposed, see, e.g., [1].

Very recently, a novel exact algorithm for solving OT, called iterated Inside Out (IIO), has been proposed [2]. The strength of this new method relies on the fact that potentially many pivoting operations are performed for each computation of dual multipliers and reduced costs. IIO requires in input a basic feasible solution and is composed by two phases that are iterated until an optimal basic feasible solution is found. The first 'inside' phase progressively improves the current basic solution by increasing the value of several non-basic variables with negative reduced cost and typically outputs a non-basic feasible solution corresponding to an interior point of the constraints' set polytope. The second 'out' phase operates in the opposite direction by iteratively setting to zero several variables until a new improved basic feasible solution is reached. The basic version of IIO combined with the shielding neighborhood [4] shows up to be approximately twice faster than [7] on the DOTmark instances.

Here, we propose a variant of IIO devised for solving much more efficiently the DOTmark instances w.r.t. [2]. The new version of IIO solves the DOTmark instances exploiting the characterizing structure of the transportation costs: these costs depend on the couple of indexes (i,j) of the related problem variables and present strong regularity in the way they increase or decrease according to a change of index i or j. Given the dual multipliers of the current basic solution, the described structure of costs enables the algorithm to predict from scratch the positivity of a large set of reduced costs correspondingly avoiding their computation. The new version of IIO computes, at each iteration, an *ad-hoc* subset of *neighbor* reduced costs that guarantees the optimality of the final solution but strongly reduces the total number of reduced costs to be computed. Let denote this neighbor subset as the DOTmark neighborhood (DM-NGH).

Computational experiments revealed DM-NGH to be very effective but characterized by a long-tail effect, namely a large subset of iterations (the final ones) showing marginal improvement of the objective function value. To overcome this phenomenon, we designed a second new neighborhood of a current basic solution as follows. For each variable $x_{i,j}$ of a DOTmark instance with image size L, the closest variables to $x_{i,j}$ in terms of transportation cost are variables $x_{i+1,j}$, $x_{i-1,j}$, $x_{i,j+1}$, $x_{i,j-1}$, $x_{i+L,j}$, $x_{i-L,j}$, $x_{i,j+L}$ and $x_{i,j-L}$. We denote by $8-set(x_{i,j})$ the corresponding subset of variables, and by 8-vars neighborhood (8V-NGH) the union of all $8-set(x_{i,j})$ subsets over all basic variables $x_{i,j}$. Note that 8V-NGH nearly always contains at least one variable with negative reduced cost. The new IIO for DOTmark instances first applies DM-NGH until the improvement of the objective function is regularly above a fixed threshold, then applies 8V-NGH until negative reduced costs are found, finally completes the optimization reapplying DM-NGH (often just to prove optimality).

In Table 1, we report a computation time comparison of three versions of IIO plus Bonneel's NS [7] on the DOTmark instances considered in [3]. All experiments ran as single thread processes on a laptop with a *11th Gen Intel Core i7-1165G7 2.80GHz* $\times$ *8* processor and 16GB of RAM, and running Ubuntu 20.04.5 LTS. Table 1 shows that the best version of IIO algorithm for the DOTmark instances strongly outperforms the current state-of-the-art approaches.

	IIO with shielding	IIO with DM-NGH	IIO with DM-NGH + 8V-NGH	Bonneel's NS
image size	[sec.]	[sec.]	[sec.]	[sec.]
32x32	0.13	0.10	**0.09**	0.22
64x64	2.41	1.45	**1.18**	5.79
128x128	175.74	41.67	**23.87**	335.24

TAB. 1: Algorithms performances on DOTmark instances

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