# Social ranking solutions and the desirability property

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## 1 Introduction

A coalitional ranking describes a total preorder over a set of coalitions, subsets of a finite set of agents. A social ranking solution is a function that, based on a coalitional ranking, produces a total preorder over the set of agents.

Social ranking solutions in the literature are proposed under careful consideration of a set of axioms they satisfy. Here, we characterize the lex-cel [2] and $L^{(1)}$ [1] solution on the basis of the *desirability property*, which has been widely studied in the field of cooperative games [3, 4]. The proposed interpretation of this axiom in an ordinal setting is as follows: given a coalitional ranking $\succsim$ of the subsets of a finite set $N$, if $S \cup \{i\} \succsim S \cup \{j\}$ for all sets $S \subseteq N \setminus \{i, j\}$ containing neither $i$ nor $j$, then the social ranking solution should rank $i$ over $j$.

## 2 Characterizing lex-cel and $L^{(1)}$ under desirability

### 2.1 Desirability

The question of which agent ranks better than the other is of particular interest in simple games. In comparing two agents, it is reasonable to expect some agent $i$ to be ranked at least as highly as another agent $j$ if every winning coalition containing $j$ without $i$ is also a winning coalition if it contains $i$ but not $j$. This *desirability relation* has been established by Isbell [3] for simple games and successively it has been studied by many authors in the more general domain of Transferable Utility games [4].

In this work, we first adapt the desirability property in the setting of the social ranking problem. We denote by $\mathcal{T}^{\mathcal{P}^N}$ the set of coalitional rankings (total preorders) of the subsets of $N$, and by $\mathcal{T}^N$ the set of rankings of the elements of $N$. A *social ranking* is a function $R : \mathcal{T}^{\mathcal{P}^N} \to \mathcal{T}^N$ associating to each coalitional ranking $\succsim \in \mathcal{T}^{\mathcal{P}^N}$ a ranking $R^{\succsim}$ of the elements of $N$. To formally introduce the desirability property for social rankings, we first need to define the notion of desirability relation. Given a set of agents $N$ and a coalitional ranking $\succsim$ of the subsets of $N$, a desirability relation $R_d^{\succsim} \subseteq N \times N$ is defined as follows:

$$i \; R_d^{\succsim} \; j \;\Leftrightarrow\; S \cup \{i\} \succsim S \cup \{j\} \text{ for all } S \subseteq N \setminus \{i, j\}.$$

**Axiom 1 (Desirability)** Let $i, j \in N$. A social ranking $R : \mathcal{T}^{\mathcal{P}^N} \to \mathcal{T}^N$ satisfies the *desirability* property if $[i \; I_d^{\succsim} \; j \;\Rightarrow\; i \; I^{\succsim} \; j]$ and $[i \; P_d^{\succsim} \; j \;\Rightarrow\; i \; P^{\succsim} \; j]$ for any $\succsim \in \mathcal{T}^{\mathcal{P}^N}$ (notations $I$ and $P$ stand for the symmetric and asymmetric part of a corresponding relation $R$, respectively).

## 2.2 Lex-cel and $L^{(1)}$ solutions

We show that the desirability property builds the foundation for characterizing at least two social rankings from the literature: lex-cel [2] and $L^{(1)}$ [1]. Both solutions put an emphasis on the number of times an agent appears in higher ranking coalitions. One important distinction between these two is the consideration of coalition sizes, which only the $L^{(1)}$ takes into account. Specifically, the lex-cel is defined as follows. Given a coalitional ranking $\succsim$ and its associated quotient ranking $\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \ldots \succ \Sigma_l$, we denote by $i_k$ the number of sets in $\Sigma_k$ containing $i$:

$$i_k = |\{S \in \Sigma_k : i \in S\}|$$

for $k = 1, \ldots, l$. Now, let $\theta_{\succsim}(i)$ be the $l$-dimensional vector $\theta_{\succsim}(i) = (i_1, \ldots, i_l)$.

**Definition 1** *The* lexicographic excellence (lex-cel) solution *is the function $R_{\boldsymbol{le}}$ defined for any ranking $\succsim \in \mathcal{T}^{\mathcal{P}^N}$ as*

$$i \; R_{\boldsymbol{le}}^{\succsim} \; j \qquad \text{if} \qquad \theta_{\succsim}(i) \;\; \geq_L \;\; \theta_{\succsim}(j).$$

*where $\geq_L$ lexicographic order among vectors: $\mathbf{i} \geq_L \mathbf{j}$ if either $\mathbf{i} = \mathbf{j}$ or $\exists s : i_k = j_k, k = 1, \ldots, s-1$ and $i_s > j_s$.*

Together with the axiom of *coalitional anonymity*, saying that only the positions of elements of $N$ in the coalitional ranking matters, and the names or number of mates does not count, and the axioms of *independence of the worst set*, stating that a strict relation in the social ranking of two elements should not be modified by a change in the ranking of the worst coalitions (see [2] for more details on these two axioms) we are able to prove the following result.

**Theorem 1** *The lex-cel solution is the unique social ranking fulfilling the axioms of desirability, coalitional anonymity, and independence of the worst set.*

# 3 Conclusions et perspectives

Desirability can be seen as an important requirement for social ranking solutions to be deemed reasonable. While the proposed solution concepts have already been analyzed, this research aims to provide further analytical depth and robustness, as well as broadening the perspective of what favorable ranking solutions look like.

For future research, it could be valuable to apply these concepts to other established or new ranking solutions. It may also provide ways to explore the scalability in larger, more complex systems. In particular, the case of incomplete information under these axioms remains largely open.

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